5-2. Multidimensional Problems Expressible in Terms of One-Dimensional Ones. Use of One-Dimensional Charts

In this section we consider a class of multidimensional problems whose solution can be found by expressing the problem in terms of two or more one-dimensional problems. First an example will be taken from two-dimensional cartesian geometry, then the results will be generalized to three-dimensional cartesian and other geometries.

Example 5-12. An infinitely long rod of rectangular cross section ($2L \times 2l$) having the uniform initial temperature T_0 is plunged suddenly into a bath at constant temperature T_∞. The heat transfer coefficient is h (Fig. 5-13). We wish to find the unsteady temperature of the rod.

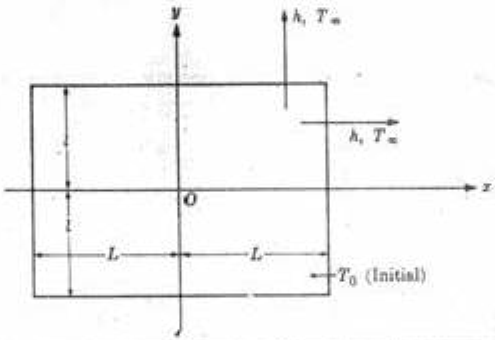

FIG. 5-13

The formulation of the problem in terms of the dimensionless temperature $\vartheta = (T - T_\infty)/(T_0 - T_\infty)$ is

$$\frac{\partial\vartheta}{\partial t} = a\left(\frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2}\right), \qquad \vartheta(x, y, 0) = 1,$$

$$\frac{\partial\vartheta(0, y, t)}{\partial x} = 0, \qquad -k\frac{\partial\vartheta(L, y, t)}{\partial x} = h\vartheta(L, y, t), \tag{5-81}$$

$$\frac{\partial\vartheta(x, 0, t)}{\partial y} = 0, \qquad -k\frac{\partial\vartheta(x, l, t)}{\partial y} = h\vartheta(x, l, t).$$

The problem could have been solved by the usual separation $\vartheta(x, y, t) = X(x)Y(y)\tau(t)$. Here, however, a less restrictive form,

$$\vartheta(x, y, t) = X(x, t)Y(y, t), \tag{5-82}$$

will be *assumed*. If Eq. (5-82) works, we are led to the important conclusion that *it is possible to express an unsteady two-dimensional problem as the product of two unsteady one-dimensional problems.*

Introducing Eq. (5-82) into the differential equation of Eq. (5-81) and rearranging gives

$$\frac{1}{X}\left(\frac{\partial X}{\partial t} - a\frac{\partial^2 X}{\partial x^2}\right) = -\frac{1}{Y}\left(\frac{\partial Y}{\partial t} - a\frac{\partial^2 Y}{\partial y^2}\right) \tag{5-83}$$

Since x and y can vary independently, both sides of Eq. (5-83) must be independent of either variable, and equal to a parameter, say $\pm\lambda^2(t)$, which now may depend on the common variable, time. However, because of the geometric as well as thermal symmetry of the problem, the characteristic-value problems in the x- and y-directions must be similar. This can occur only with $\lambda^2(t) = 0$. Employing this value of $\lambda^2(t)$, and introducing Eq. (5-82) into the initial and boundary conditions of Eq. (5-81), we have

$$\frac{\partial X}{\partial t} = a\frac{\partial^2 X}{\partial x^2}, \qquad \frac{\partial Y}{\partial t} = a\frac{\partial^2 Y}{\partial y^2},$$

$$X(x, 0) = 1, \qquad Y(y, 0) = 1,$$

$$\frac{\partial X(0, t)}{\partial x} = 0, \qquad \frac{\partial Y(0, t)}{\partial y} = 0,$$

$$-k\frac{\partial X(L, t)}{\partial x} = hX(L, t), \qquad -k\frac{\partial Y(l, t)}{\partial y} = hY(l, t).$$

Thus the problem becomes expressible as the product of two one-dimensional unsteady problems. These are identical to each other, and to the formulation of Example 5-3, whose solution is given by Eq. (5-13). The dimensionless form of Eq. (5-13) is

$$\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L \text{ or } 2l \\ \text{Plate}}} = 2\sum_{n=1}^{\infty}\left(\frac{\sin\mu_n}{\mu_n + \sin\mu_n\cos\mu_n}\right)e^{-\mu_n^2 \mathrm{Fo}}\cos\mu_n\xi, \tag{5-84}$$

where $\xi = x/L$ (or y/l), $\mathrm{Fo} = at/L^2$ (Fourier number), $\mu_n = \lambda_n L$, and μ_n are the zeros of $\mu_n \sin\mu_n = \mathrm{Bi}\cos\mu_n$. In Fig. 5-14,* $[(T - T_\infty)/(T_0 - T_\infty)]_{2L\text{ Plate}}$ is plotted against Fo for the values $\xi = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$, with Bi as parameter.

Thus, noting that Eq. (5-82) may be written in the form

$$\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L, 2l \\ \text{Rod}}} = \left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L \\ \text{Plate}}}\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2l \\ \text{Plate}}} \tag{5-87}$$

and, using Eq. (5-87) with the one-dimensional temperature charts given by Fig. 5-14, we may readily find the instantaneous temperature of an infinitely long rod of rectangular cross section ($2L \times 2l$).

The foregoing procedure may now be extended to three-dimensional cartesian and two-dimensional cylindrical geometries. The result for the cartesian case is

$$\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L, 2L, 2l \\ \text{Parallelepiped}}} = \left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L \\ \text{Plate}}}\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L \\ \text{Plate}}}\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2l \\ \text{Plate}}}$$

and that for a cylindrical rod of radius R and height $2L$ is

$$\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2R, 2L \\ \text{Rod}}} = \left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{\text{Infinite} \\ 2R\text{ Rod}}}\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)_{\substack{2L \\ \text{Plate}}} \tag{5-89}$$